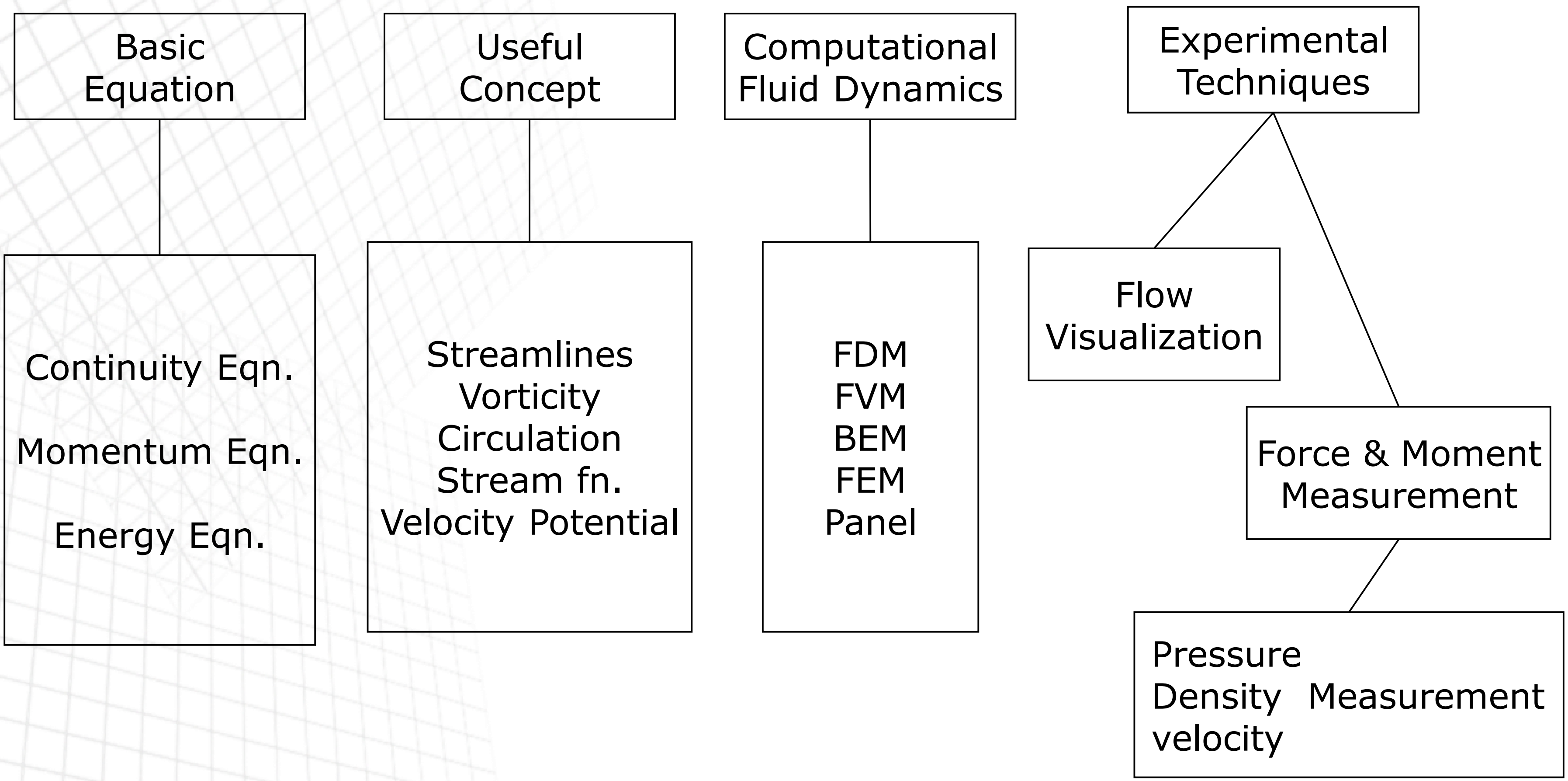
< Aerodynamic Tools >



< 2.1. Vector Relations >

$$\vec{A} + \vec{B} = \vec{C}$$

Dot Product (Scalar Product) $\vec{A} \bullet \vec{B} = |A||B|\cos\theta$

Cross Product (Vector Product) $\vec{A} \times \vec{B} = (|A||B|\sin\theta)\vec{e} \quad (ringt - hand\, rule)$

Orthogonal Coordinate

* Cartesian coordinate

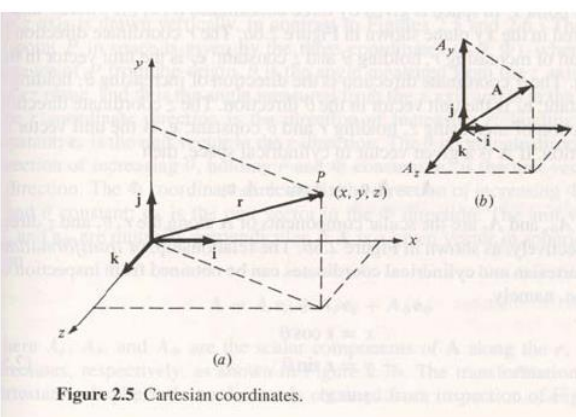
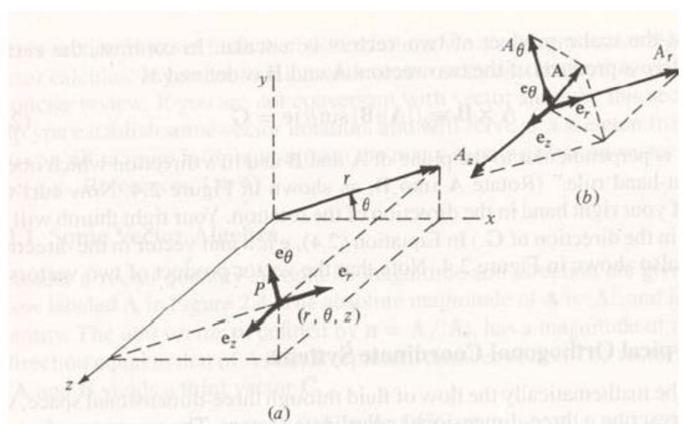


Figure 2.5 Cartesian coordinates.

* Cylindrical coordinate



* Spherical coordinate

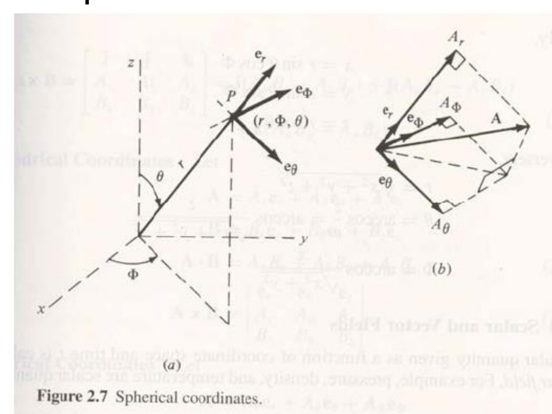


Figure 2.7 Spherical coordinates.

< 2.2. Scalar & Vector Fields >

* **Scalar quantities**

$$p = p(x, y, z, t)$$
$$\rho = \rho(x, y, z, t)$$
$$t = t(x, y, z, t)$$

* **Vector quantities**

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

* **Products**

$$\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$$
$$\vec{B} = B_1 \vec{i} + B_2 \vec{j} + B_3 \vec{k}$$
$$\vec{A} \bullet \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$
$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \vec{i}(A_2 B_3 - A_3 B_2) + \vec{j}(A_3 B_1 - A_1 B_3) + \vec{k}(A_1 B_2 - A_2 B_1)$$

< 2.2. Scalar & Vector Fields >

*** Gradient**

$$\nabla p = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}$$

*** Divergence : rate of volume change**

$$\nabla \bullet \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \bullet \left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

*** Curl : rate of change of fluid element**

$$\nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ u & v & w \end{vmatrix}$$

< 2.2. Scalar & Vector Fields >

* **Line integral**

$$\oint_a^b \vec{A} \bullet \vec{ds} \quad conunter\ clockwise \Rightarrow 'positive'$$

* **Surface integral**

$$\oiint_s p\,\vec{ds} \rightarrow \overrightarrow{vector}$$

$$\oiint_s \vec{A} \bullet \vec{ds} \rightarrow scalar$$

$$\oiint_s \vec{A} \times \vec{ds} \rightarrow \overrightarrow{vector}$$

* **Volume integral**

$$\iiint_V \rho\, dv \rightarrow scalar$$

$$\iiint_V \vec{A}\, dv \rightarrow \overrightarrow{vector}$$

< 2.2. Scalar & Vector Fields >

* **Relation between line, surface, and volume integral**

* **Stokes Theorem**

$$\oint_c \vec{A} \bullet \vec{ds} = \iint_s (\nabla \times \vec{A}) \bullet \vec{ds} \qquad \text{(s : line)}$$

* **Divergence Theorem**

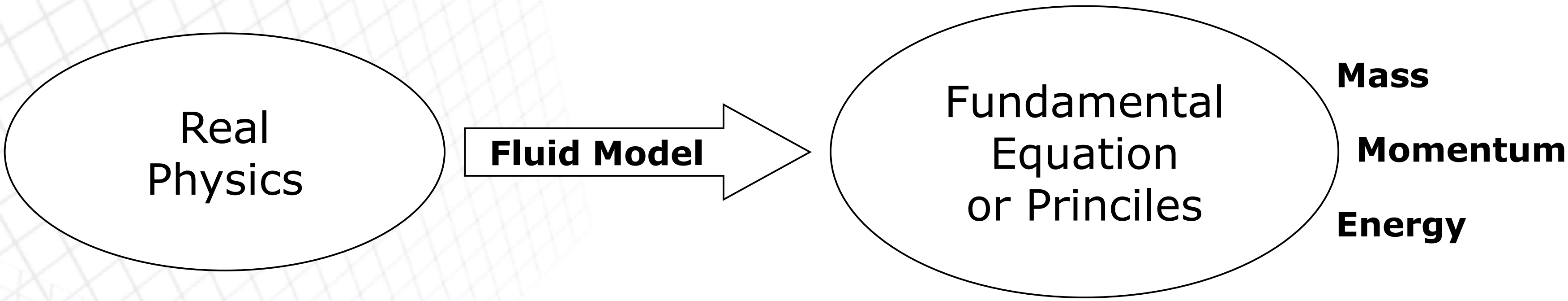
$$\oiint_s \vec{A} \bullet \vec{ds} = \iiint_v (\nabla \bullet \vec{A}) dv \qquad \text{(s : area)}$$

* **Gradient Theorem**

$$\oiint_s p\, dS = \iiint \nabla p\, dV$$

* **Control Volume** vs **Material Volume (or Control Mass)**

< 2.3. Models of the fluid >



- Conservation of mass
- Conservation of momentum
- Conservation of energy

< 2.3. Models of the fluid >

❖ Fluid Model

- **Finite Control Volume** – fixed with space (Eulerian Description)
- **Finite Material Volume** – moving with fluid (Lagrangian Description)
- **Infinitesimal Fluid Element**
- **Molecular approach** – statistical view, Boltzmann Equation

< 2.3. Models of the fluid >

❖ Physical Meaning of Divergence of Velocity

- The total change of the whole control volume over time Δt

$$\triangle \overline{V} = \oiint (\vec{v} \triangle t) \bullet \vec{ds}$$

$$\frac{D\overline{V}}{Dt} = \frac{\triangle \overline{V}}{\triangle t} = \oiint_s \vec{v} \bullet \vec{ds} = \iiint_V (\nabla \bullet \vec{v}) dV$$

- Think an infinitesimal volume δv

$$\frac{D(\delta V)}{Dt} = \iiint_{\delta V} (\nabla \bullet \vec{V}) dV \approx (\nabla \bullet \vec{V}) \delta V$$

$$\therefore \nabla \bullet \vec{V} = \frac{1}{\delta V} \frac{D(\delta V)}{Dt}$$

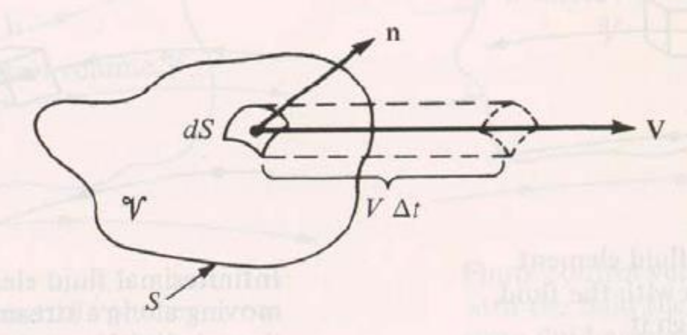


Figure 2.15 Moving control volume used for the physical interpretation of the divergence of velocity.

- $\nabla \bullet \vec{V}$ is the time rate of change of the volume of a moving fluid element

< 2.4. Continuity equation >

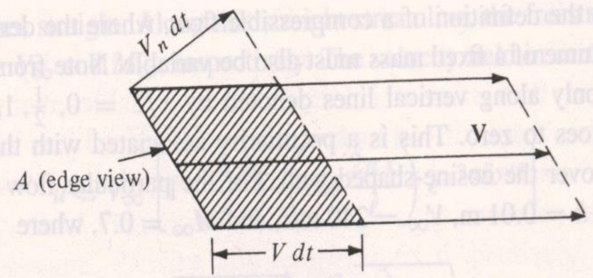


Figure 2.18 Sketch for discussion of mass flow through area A in a flow field.

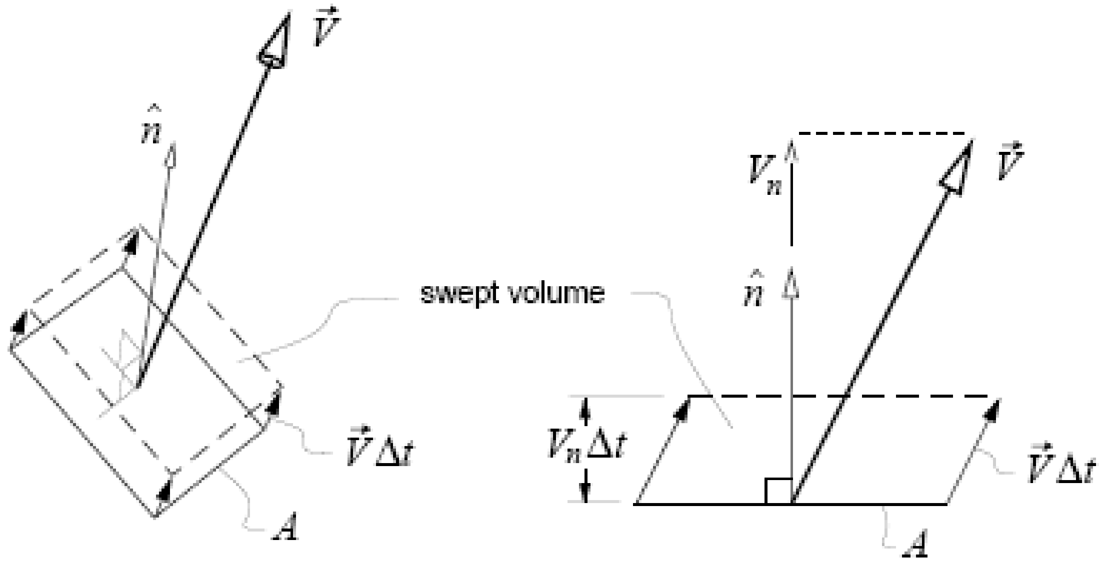
$$volume = (v_n dt)A$$
$$mass = \rho(v_n dt)A$$
$$mass\ flowrate = \dot{m} = \frac{\rho A v_n dt}{dt} = \rho A v_n$$
$$mass\ flux = \frac{\dot{m}}{A} = \rho v_n \quad (unit: kg/sm^2)$$

< 2.4. Continuity equation >

❖ Mass flow

- The plane of fluid particles which are on the surface at time *t* will move off the surface at time $t+\Delta t$, sweeping out a volume given by $\Delta v = V_n A \Delta t$.

(where $V_n = \boldsymbol{V} \cdot \boldsymbol{n}$)

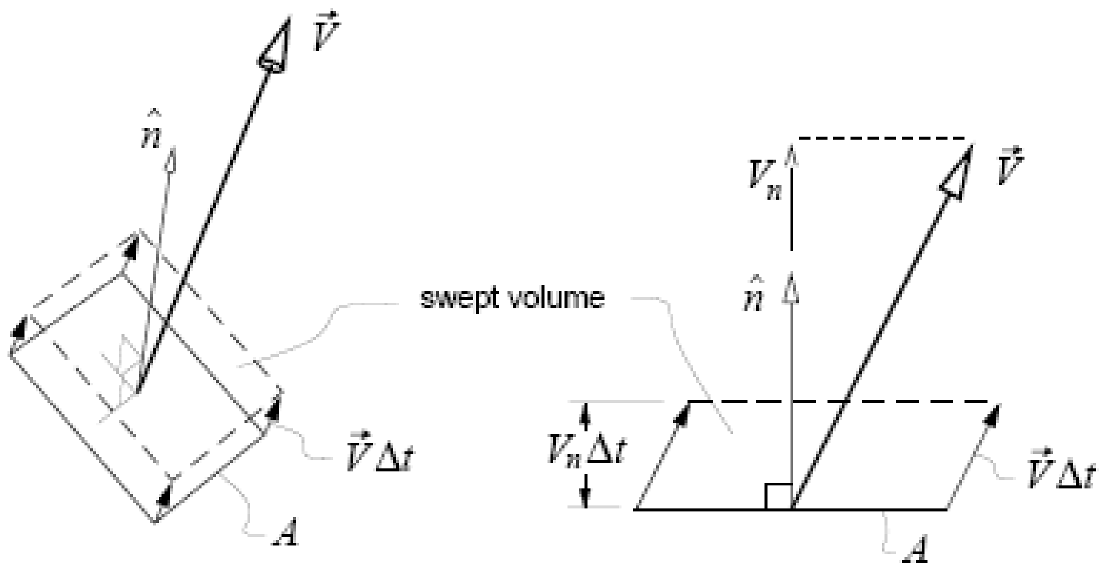


< 2.4. Continuity equation >

❖ Mass flow

- The mass of fluid in this swept volume, which evidently passed through the area during the Δt interval, is

$$\Delta m = \rho \Delta v = \rho V_n A \Delta t$$



< 2.4. Continuity equation >

❖ Mass flow

- The *mass flow* is defined as the time rate of this mass passing though the area.

$$mass\ flow = \dot{m} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho V_n A$$

- The *mass flux* is defined simply as mass flow per area.

$$mass\ flux = \frac{\dot{m}}{A} = \rho V_n$$

< 2.4. Continuity equation >

❖ Principle #1 : Mass should be conserved

- Consider a control volume,

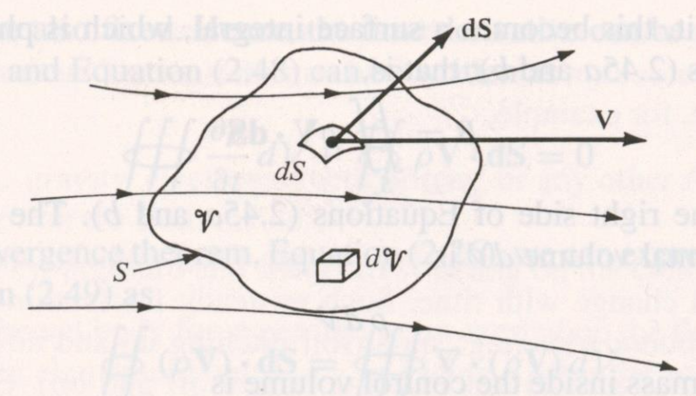


Figure 2.19 Finite control volume fixed in space.

Net increase in the contorl volume = In - Out

< 2.4. Continuity equation >

❖ Principle #1 : Mass should be conserved

- Increase of mass in CV

$$\frac{\partial}{\partial t}\oiiint S\,dV$$

- Net flow = outflow - inflow

$$\oiint \rho\vec{v}\,d\vec{s}$$

$$\boxed{\frac{\partial}{\partial t}\oiiint S\,dV+\oiint \rho\vec{v}\,d\vec{s}=0 \quad \text{integral form of continuity}}$$

< 2.4. Continuity equation >

❖ Principle #1 : Mass should be conserved

$$\frac{\partial}{\partial t}\oiiint S\,dV + \oiint \rho\vec{v}\,d\vec{s} = 0 \quad \text{integral form of continuity}$$

* **fixed volume**

$$\Rightarrow \oiiint \frac{\partial \rho}{\partial t} dV + \oiint \rho\vec{v}\,d\vec{s} = 0$$

* **divergence theorem**

$$\Rightarrow \oiint (\rho\vec{v})\,d\vec{s} = \oiiint \nabla \bullet (\rho\vec{v})\,dV$$

$$\Rightarrow \oiiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\vec{v})\right] dV = 0$$

⇓ **To satisfy with arbitrary control volume**

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\vec{v}) = 0 \quad \textit{at a point the flow}$$

< 2.4. Continuity equation >

❖ Principle #1 : Mass should be conserved

$$\text{If steady, } \frac{\partial}{\partial t} = 0 \Rightarrow \rho = \rho(x, y, z) \text{ space only!!!}$$

$$\cancelto{0}{\frac{\partial}{\partial t} \oiiint_V s\, dV} + \oiint_S \rho \vec{v}\, d\vec{s} = 0$$

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \nabla \bullet (\rho \vec{v}) = 0$$

< 2.5. Momentum equations >

❖ **Principle #2 :**

Time rate of momentum change = Force

- Rate of momentum change inside the control volume

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV$$

- Net flow of momentum = $(\ )_{in} - (\ )_{out}$

$$\iint_S (\rho\underbrace{\vec{v}\bullet d\vec{S}}_{\dot{m}})\vec{v}$$

< 2.5. Momentum equations >

- Force acting on the control volume
  - Body force : acting on the body
    gravity, electromagnetic force
  - Surface force : shear stress (←due to viscosity)

$$\frac{\partial}{\partial t}\oiiint_V \rho\vec{v}\,dV + \oiint_S (\rho\vec{v}\,d\vec{S})\vec{v} = -\oiint_S p\,d\vec{s} + \oiiint_V \rho\vec{f}\,dV + F_{viscous}$$

Opposite direction to the surface

*** Integral Form of Momentum Equation**

< 2.5. Momentum equations >

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV + \oiint_S (\rho\vec{v}\,d\vec{S})\vec{v} = -\oiint_S p\,d\vec{s} + \iiint_V \rho\vec{f}\,dV + F_{viscous}$$

- Gradient Theorem $\Longrightarrow$ $-\oiint_S p\,d\vec{s} = -\iiint_V \nabla p\,dV$
- Divergence Theorem $\Longrightarrow$ $\oiint_S (\rho\vec{v}\,d\vec{S})\vec{v} = \iiint \nabla(\rho\vec{v}\vec{v})\,dV$

$$\Longrightarrow \iiint_V \left[\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\bullet(\rho\vec{v}\vec{v})\right]dV = \iiint_V \left[-\nabla p + \rho\vec{f} + \vec{F}_{viscous}\right]$$

$$\boxed{\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\bullet(\rho\vec{v}\,\vec{v}) = -\nabla p + \rho\vec{f} + \overrightarrow{F_{viscous}}}$$

* **Differential Form of Momentum Equation**

< 2.5. Momentum equations >

$$\boxed{\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla \bullet (\rho\vec{v}\,\vec{v}) = -\nabla p + \rho\vec{f} + \overrightarrow{F_{viscous}}}$$

*** dyadic (momentum flux tensor)**

- Divergence of a dyadic becomes a vector using the relation of

$$\nabla \bullet (\rho\vec{v}\,\vec{v}) = \rho(\vec{v} \bullet grad)\vec{v} + \vec{v}\nabla \bullet (\rho\vec{v})$$

$$\vec{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v} \bullet grad\vec{v} + \vec{v}\nabla \bullet (\rho\vec{v}) = -\nabla\rho + \rho\vec{f} + \vec{F}_{viscous}$$

$$\frac{\partial\rho}{\partial t} = -\nabla \bullet (\rho\vec{v}) \quad \leftarrow continuity$$

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v} \bullet grad)\vec{v} = -\nabla p + \rho\vec{f} + \vec{F}_{viscous}$$

*** u-direction :** $\rho\frac{\partial u}{\partial t} + \rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \rho f_x + F_{viscous}$

< 2.7. Energy equation >

❖ Principle : Energy is conserved

- Gibb's Equation

$$\delta q + \delta w = de$$

Rate of heat added to fluid inside CV from surrounding

Rate of work done on fluid inside control volue

Rate of change of Energy of fluid as it flows through CV

< 2.7. Energy equation >

$$\boxed{\dot{Q} + \dot{W} = \dot{E}}$$

$$\dot{Q} : \iiint_V \dot{q}\rho\, dV + \dot{Q}_{viscous}$$

Viscous heat addition

Rate of volumetric heating

$$\dot{W} : -\oiint (P\,ds) \bullet \vec{v} + \iiint_V (\rho \vec{f}\, dV) \bullet \vec{v} + \dot{w}_{viscous}$$

Pressure term

Body force term

Viscous work term

$$\dot{E} : \frac{\partial}{\partial t} \iiint_V \rho\left(e + \frac{v^2}{2}\right) \delta V + \oiint_s \rho\left(e + \frac{v^2}{2}\right) \vec{v} \bullet \vec{ds}$$

Time rate of change of total energy inside CV

Net rate of change of total energy across CV

< 2.7. Energy equation >

* **Integral Form of energy equation**

$$\oiiint_V \dot{q}\rho dV + \dot{Q}_{viscous} - \oiint (p d\vec{s}) + \oiiint (\rho \vec{f} dV)\vec{v} + \dot{W}_{viscous}$$

$$= \frac{\partial}{\partial t} \oiiint_V \rho(e + \frac{v^2}{2}) dV + \oiint_s \rho(e + \frac{v^2}{2}) \vec{v}\, d\vec{s}$$

* **Differential Form of energy equation**

$$\frac{\partial}{\partial t}[\rho(e + \frac{v^2}{2})] + \nabla \bullet [\rho(e + \frac{v^2}{2})\vec{v}]$$

$$= \rho\dot{q} - \nabla(\rho\vec{v}) + \rho\vec{f} \bullet \vec{v} + \dot{Q}'_{viscous} + \dot{W}'_{viscous}$$

< 2.7. Energy equation >

* **Differential Form of energy equation**

$$\frac{\partial}{\partial t}[\rho(e+\frac{v^2}{2})]+\nabla \bullet [\rho(e+\frac{v^2}{2})\vec{v}]$$
$$=\rho\dot{q}-\nabla(\rho\vec{v})+\rho\vec{f}\bullet\vec{v}+\dot{Q}'_{viscous}+\dot{W}'_{viscous}$$

If steady,

$$\Rightarrow\frac{\partial}{\partial t}(\ )=0,\ \dot{Q}=\dot{W}=0\,,\dot{q}=0$$

$$\oiint_s \rho(e+\frac{v^2}{2})\vec{v}\bullet\vec{ds}=-\oiint(p\vec{ds})\bullet\vec{v}$$

$$\nabla\bullet(\rho(e+\frac{v^2}{2})\vec{v})=-\nabla\bullet(p\vec{v})$$

Fundamental Principles & Equations

- **Now** <u>**6 unknowns**</u> <u>**5 equations +1**</u>

6 unknowns	5 equations +1
ρ	**Continuity equation**
$\vec{v}(x,y,z)$	**Momentum equation**
$e = c_v T$	**Energy equation**
p	**Equation of state** $p = \rho R T$